# Hyperspheres and control of spin chains

*Alexander Yu. Vlasov*
Federal Radiological Center (IRH), 197101 Mira Street 8, St.–Petersburg, Russia

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**Abstract**

Here is considered application of Spin($m$) groups in theory of quantum control of chain with spin-$\frac{1}{2}$ systems. It may be also compared with $m$-dimensional analogues of Bloch sphere, but has nontrivial distinctions for chain with more than one spin system.

## 1 Introduction

It is convenient to use sphere to represent state of one spin-1/2 system. It is so called Bloch or Poincaré sphere. The state of space of $n$ spin-1/2 systems is complex Hilbert space with dimension $2^n$, but due to normalization condition there are $2^n - 1$ complex or $2\,(2^n-1)$ real parameters. So for $n = 1$ there are $2 = 2(2^1-1)$ real parameters, *e.g.* two Euler angles describing point on surface of sphere. Any unitary transformation of spin-1/2 system corresponds to rotation of Bloch sphere in agreement with $2 \to 1$ isomorphism of groups SU(2) and SO(3).

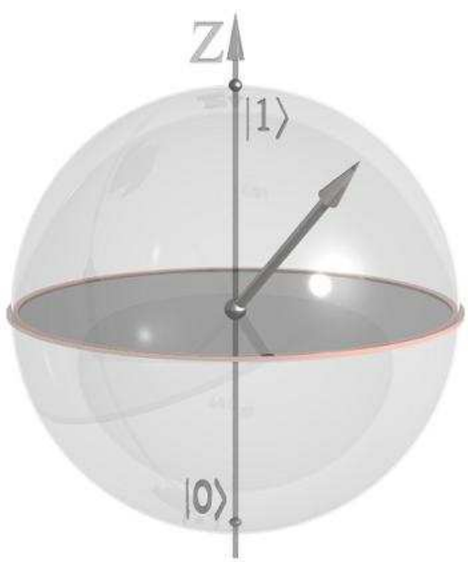


Figure 1: Bloch sphere.

For $n > 1$ there is no such convenient visualization of space of *all* states with higher dimensional spheres[1], but exist some interesting subspaces with such property. The subspaces are important for theory of quantum computations and control, because for some physical systems they may correspond to *simpler accessible* set of physical states.

## 2 Rotations and Spin groups

Let us consider for chain with $n$ qubits set with $2n$ Hermitian matrices

$$\begin{aligned} \boldsymbol{e}_{2k} &= \underbrace{\boldsymbol{\sigma}_z \otimes \cdots \otimes \boldsymbol{\sigma}_z}_{k} \otimes \boldsymbol{\sigma}_x \otimes \underbrace{\mathbf{1} \otimes \cdots \otimes \mathbf{1}}_{n-k-1}, \\ \boldsymbol{e}_{2k+1} &= \underbrace{\boldsymbol{\sigma}_z \otimes \cdots \otimes \boldsymbol{\sigma}_z}_{k} \otimes \boldsymbol{\sigma}_y \otimes \underbrace{\mathbf{1} \otimes \cdots \otimes \mathbf{1}}_{n-k-1}. \end{aligned} \tag{1}$$

The set is well known in quantum mechanics due to Jordan, Wigner and Weyl works [1], because operators

$$\boldsymbol{a}_k = \frac{\boldsymbol{e}_{2k} + i\boldsymbol{e}_{2k+1}}{2}, \quad \boldsymbol{a}_k^\dagger = \frac{\boldsymbol{e}_{2k} - i\boldsymbol{e}_{2k+1}}{2} \tag{2}$$

(*i.e.*, $2^n \times 2^n$ complex matrices) provide representation of *canonical anticommuting relations* (CAR)

$$\{\boldsymbol{a}_k, \boldsymbol{a}_j\} = \{\boldsymbol{a}_k^\dagger, \boldsymbol{a}_j^\dagger\} = 0, \quad \{\boldsymbol{a}_k, \boldsymbol{a}_j^\dagger\} = \delta_{kj}. \tag{3}$$

On the other hand, Eq. (1) may be used for construction Spin($2n$) and Spin($2n+1$) groups [2]. The Spin($2n+1$) groups has $2 \to 1$ isomorphism with group of rotations SO($2n+1$) and so for $n = 1$ we have usual model with Bloch sphere and group SO(3) of three-dimensional rotations of the sphere.

[1] But see **Note** on page 3.

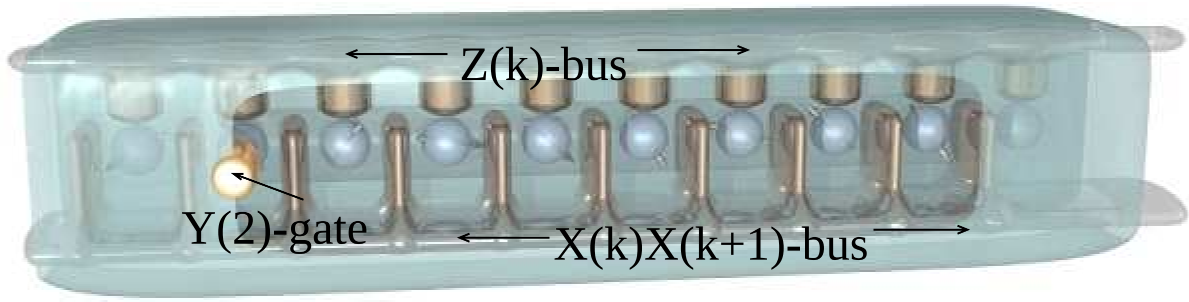


Figure 2: Scheme of spin chain with control buses.

For $n > 1$ it is also possible to consider groups $\mathrm{Spin}(2n+1)$ and $\mathrm{SO}(2n+1)$, but they may not describe all possible transformations of system with $n$ qubits. Such transformations may be described by huge group $\mathrm{SU}(2^n)$ with dimension $4^n - 1$, but group $\mathrm{SO}(2n+1)$ has dimension $(2n+1)n$ and only for $n = 1$ both numbers coinside:

| $n$ | 1 | 2 | 3 | 4 | 5 | 10 |
|---|---|---|---|---|---|---|
| $(2n+1)n$ | 3 | 10 | 21 | 36 | 55 | 210 |
| $4^n - 1$ | 3 | 15 | 63 | 255 | 1023 | 1048575 |

The Eq. (1) are Hermitian matrices and may be considered as set of $2n$ Hamiltonians for system with $n$ qubits. If to use *only* these Hamiltonians for control of system, then unitary evolution belong only some subgroup of $S_o \subset \mathrm{SU}(2^n)$, *i.e.*, the control is *not universal*. Despite the subgroup $S_o$ belongs to such exponentially big space, it is ismorphic to $\mathrm{Spin}(2n+1)$ [3] and due to usual relation of Spin groups with rotations of $2n+1$-dimensional hypersphere may be considered as higher dimensional analogue model of Bloch sphere rotations.

It should be mentioned, that together with $S_o \subset \mathrm{SU}(2^n)$, $S_o \cong \mathrm{Spin}(2n+1)$, it is also useful to consider (maybe more familiar) *even* subgroup $S_e \subset S_o$, $S_e \cong \mathrm{Spin}(2n)$ [2, 3]. The subgroup is generated by even elements $\boldsymbol{d}_k = i\boldsymbol{e}_k\boldsymbol{e}_{k+1}$

$$\boldsymbol{d}_{2k} = \underbrace{\mathbf{1}\otimes\cdots\otimes\mathbf{1}}_{k} \otimes \boldsymbol{\sigma}_z \otimes \underbrace{\mathbf{1}\otimes\cdots\otimes\mathbf{1}}_{n-k-1}, \tag{4a}$$

$$\boldsymbol{d}_{2k+1} = \underbrace{\mathbf{1}\otimes\cdots\otimes\mathbf{1}}_{k} \otimes \boldsymbol{\sigma}_x \otimes \boldsymbol{\sigma}_x \otimes \underbrace{\mathbf{1}\otimes\cdots\otimes\mathbf{1}}_{n-k-2}. \tag{4b}$$

Now it is possible to add any operator $\boldsymbol{e}_k$, say

$$\boldsymbol{e}_0 = \boldsymbol{\sigma}_x \otimes \underbrace{\mathbf{1}\otimes\cdots\otimes\mathbf{1}}_{n-1}, \tag{5}$$

to provide control on $S_o$ and it was shown in [3], that it is enough to add also any *third (or fourth) order* operator like $\boldsymbol{e}_k\boldsymbol{e}_l\boldsymbol{e}_m$, say

$$\boldsymbol{e}_0\boldsymbol{e}_1\boldsymbol{e}_3 = \mathbf{1} \otimes \boldsymbol{\sigma}_y \otimes \underbrace{\mathbf{1}\otimes\cdots\otimes\mathbf{1}}_{n-2}, \tag{6}$$

to provide universal control, whole group $\mathrm{SU}(2^n)$.

The subgroup $S_e$ also may be generated by Hermitian bilinear combinations of fermionic annihilation and creation operators Eq. (2), *i.e.*

$$\boldsymbol{a}_j\boldsymbol{a}_k^\dagger + \boldsymbol{a}_k\boldsymbol{a}_j^\dagger, \quad \boldsymbol{a}_j\boldsymbol{a}_k + \boldsymbol{a}_k^\dagger\boldsymbol{a}_j^\dagger. \tag{7}$$

It should be mentioned, that Eq. (2) and Eq. (7) here should be considered rather from point of view of *simulation* of quantum control and computations with fermionic systems [4, 5], because most methods used above may be applied to arbitrary system of $n$ qubits and are not related directly with fermionic statistic of particles in spin chain.

The note about simulation may be quite essential, say in usual fermionic systems [5] as well as in *linear optics* KLM [6] model of quantum computing (close related with the fermionic operators [7]) appearance of the group $S_e$ generated by bilinear combinations produces specific difficulty. Really, universal control suggest exponentially big space of parameters ($4^n$), but group $S_e$ has dimension only quadratic with respect to number of systems ($\dim S_e = 2n^2 - n < \dim S_o = 2n^2 + n \ll \dim \mathrm{SU}(2^n) = 4^n - 1$).

For spin chain considered here the problem with non-universality is not so essential, because it is enough to use Hamiltonians Eq. (5) and Eq. (6) to extend group $S_e$ to exponentially big group of universal control. And the two extra Hamiltonian are simply one-qubit rotations $\boldsymbol{\sigma}_x$ and $\boldsymbol{\sigma}_y$ with first and second qubit and so complexity of realization for such operations may not exceed analogous operations $\boldsymbol{d}_{2k}$ with Hamiltonian $\boldsymbol{\sigma}_z$ Eq. (4a).

On the other hand, in fermionic computations and linear optics, analogues of operators with higher order like Eq. (6) may not be realized so simply because need physical processes with very law amplitude, like nonlinear $\gamma + \gamma$ interactions. So spin chain not only may simulate some fermionic or optic computations and control available for modern state of technologies, but also some currently inaccessible, very weak processes.

Using modern jargon whole universal set of gates considered above may be denoted

$$\begin{array}{lll} \mathbf{I}) & \boldsymbol{Z}(k+1) & = \boldsymbol{d}_{2k}, \\ \mathbf{II}) & \boldsymbol{X}(k+1)\boldsymbol{X}(k+2) & = \boldsymbol{d}_{2k+1} \\ & \boldsymbol{X}(1) & = \boldsymbol{e}_0 \\ \mathbf{III}) & \boldsymbol{Y}(2) & = \boldsymbol{e}_0\boldsymbol{e}_1\boldsymbol{e}_3, \end{array} \tag{8}$$

(see Fig. 2), where gate $\boldsymbol{X}(1)$ for simplicity considered as part of $\boldsymbol{X}(k+1)\boldsymbol{X}(k+2)$-bus. In such a case buses **I** and **II** with $2n$ gates generate subgroup $S_o$ and may be associated with control of rotations in dimension $D = 2n+1$.

Such method let us not only model some processes in linear optics and fermionic systems, but also decomposes difficult task of control with exponentially big group $\mathrm{SU}(2^n)$ on simpler task of control with groups $\mathrm{SO}(2n)$ or $\mathrm{SO}(2n+1)$ and $\mathrm{SO}(2)$.

Let us consider structure of group $S_o$ with more details. Unitary matrices from groups $S_o$ and $S_e$ may be represented as

$$\Bigl\{U \in S_e \Bigm| U = \exp\bigl(\sum_{j=0}^{2n-1}\sum_{k=0}^{j-1} b_{kj}\boldsymbol{e}_k\boldsymbol{e}_j\bigr)\Bigr\}, \tag{9}$$

$$\Bigl\{U \in S_o \Bigm| U = \exp\bigl(i\sum_{j=0}^{2n-1} b_j\boldsymbol{e}_j + \sum_{\substack{j,k=0\\k<j}}^{2n-1} b_{kj}\boldsymbol{e}_k\boldsymbol{e}_j\bigr)\Bigr\}. \tag{10}$$

The number of parameters is in agreement with dimensions mentioned above $\dim S_e = 2n^2 - n$, $\dim S_o = 2n^2 + n$.

On the other hand, due to general defininion of group for any $U_1, U_2 \in S_o$ product $U_1U_2 \in S_o$, but $i\boldsymbol{e}_k = \exp(i\frac{\pi}{2}\boldsymbol{e}_k) \in S_o$ and so $2n$ elements $\boldsymbol{e}_k$ and all $2^{2n}$ possible products of the elements (up to insignificant multiplier $i$) belong to $S_o$.

The $2^{2n}$ products of $\boldsymbol{e}_k$ are simply $4^n$ possible *tensor products* with $n$ terms. Each term is either Pauli martix or $2\times 2$ unit matrix. It is discrete *Pauli group*, widely used in theory of quantum computations [8].

So the discrete Pauli group with $4^n$ elements is subgroup of continuous group $S_o$. The property show, that $S_o$ has rather nontrivial structure, because matrices from Pauli group are *basis* in $4^n$ dimensional space of matrices and so *convex hull* of $S_o$ is also $4^n$-dimensional.

Anyway, $S_o$ may be described as subspace of $\mathrm{SU}(2^n)$ isomorphic to $\mathrm{SO}(2n+1)$ and so quantum control with exponentially big space of parameters may be considered as alternating of control over "winding" subspace $S_o$ isomorphic to $\mathrm{SO}(2n+1)$ and one-parametric rotations $e^{i\alpha\boldsymbol{Y}(2)}$. The control over $S_o$ represented on Fig. 2 by two buses and one-parametric rotations as one "exceptional" gate.

**Note (June 2005)** It should be mentioned a specific case, corresponding to two qubits and not discussed in presented paper. Due to 'sporadic' isomorphism

$$\mathrm{SU}(4) \cong \mathrm{Spin}(6)$$

transformations of two qubits may be associated with rotations of 6D sphere and so it is also relevant to considered theme. It may be described using so-called *Klein correspondence* and *Plücker coordinates* (introduced first in few papers written between 1865 and 1870 by F. Klein and J. Plücker, see R. Penrose and W. Rindler, *Spinors and Space-Time,* **vol.2**, *Spinor and Twistor Methods in Space-Time Geometry*, Cambridge Univ. Press 1986), but it should be discussed elsewhere.

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